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FRESNEL'S THEORY OF DOUBLE REFRACTION

by

W. Steadman Aldis

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W. STEADMAN ALDIS, M.A.
TRINITY COLLEGE, CAMBRIDGE.

SECOND EDITION.

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A CHAPTER

ON

FRESNEL'S THEORY OF DOUBLE REFRACTION.

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BY

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Mr GRIFFIN'S tract on Double Refraction has been for some time quite out of print. The following pages are published with a view to supply the deficiency thus caused. It is hoped that they may serve as a useful companion to the latter part of the Astronomer Royal's treatise on the Undulatory Theory of Light.

W. S. A.

CAMBRIDGE,
April, 1870.

Preface to the Second Edition.

THIS Second Edition is an almost verbatim reprint of the first. It was the author's intention to incorporate this chapter in a larger work on the Wave Theory of Light. Unexpected hindrances have delayed the progress of this work beyond his expectation, and this reprint is therefore issued to satisfy the demand of present students.

NEWCASTLE-UPON-TYNE,
April, 1879.

FRESNEL'S

THEORY OF DOUBLE REFRACTION.

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1. FRESNEL'S Theory of Double Refraction supposes that the phenomena of light are produced by the vibrations of particles of ether under the influence of their mutual attractions.

The hypothesis is first made that the particles of ether are arranged in such a manner that each of them is in stable equilibrium under the influence of the attractions of the others. Let $-R$ be the potential of all the system of particles with respect to a point. Then the resolved parts of the force on the particle at this point parallel to axes arbitrarily assumed will be, if x, y, z be the co-ordinates of the point, $\dfrac{dR}{dx}, \dfrac{dR}{dy}, \dfrac{dR}{dz}$ respectively, tending towards the origin. Hence we have

$$\frac{dR}{dx}=0,\ \frac{dR}{dy}=0,\ \frac{dR}{dz}=0 \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Let the single particle at x, y, z be displaced to a point

$$x+u,\ y+v,\ z+w,$$

while all the other particles remain at rest. Then if we suppose u, v, w so small that we may neglect their squares and higher powers, the force on this displaced particle parallel to the axes will be

$$\frac{dR}{dx}+u\frac{d^2R}{dx^2}+v\frac{d^2R}{dx\,dy}+w\frac{d^2R}{dz\,dx}$$

$$\frac{dR}{dy}+u\frac{d^2R}{dx\,dy}+v\frac{d^2R}{dy^2}+w\frac{d^2R}{dy\,dz}$$

$$\frac{dR}{dz}+u\frac{d^2R}{dz\,dx}+v\frac{d^2R}{dy\,dz}+w\frac{d^2R}{dz^2}$$

respectively.

Of these the first term in each vanishes by (1), and putting

$$\frac{d^2R}{dx^2}=A,\ \frac{d^2R}{dy^2}=B,\ \frac{d^2R}{dz^2}=C,\ \frac{d^2R}{dy\,dz}=A',\ \frac{d^2R}{dz\,dx}=B',\ \frac{d^2R}{dx\,dy}=C'$$

we get, if X, Y, Z denote the forces parallel to the axes on the displaced particle,

$$\left.\begin{aligned} X &= Au + C'v + B'w \\ Y &= C'u + Bv + A'w \\ Z &= B'u + A'v + Cw \end{aligned}\right\} \quad \text{..............(2).}$$

Now if we construct the quadric whose equation is

$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy = 1 \ \ldots \ (3)$$

the direction of the resultant force whose components are X, Y, Z is perpendicular to the plane which bisects all chords of the surface (3) parallel to the direction of displacement of the particle; for the equation of this plane is

$$\xi\,(Au + C'v + B'w) + \eta\,(C'u + Bv + A'w) + \zeta\,(B'u + A'v + Cw) = 0.$$

The resultant force on the particle will therefore not usually coincide with the direction of its displacement; and if we suppose the particle free to move under the action of this force it will not usually return to its old position. There will be however three directions of displacement with which the directions of the force of restitution will coincide, namely the directions of the three principal axes of the surface (3).

If these directions be taken as axes of co-ordinates the equation (3) reduces to

$$Ax^2 + By^2 + Cz^2 = 1,$$

and the equations (2) reduce to

$$X = Au,\quad Y = Bv,\quad Z = Cw.$$

Now it is evident that if u, v, w are all positive, X, Y, Z must all tend towards the origin, since the equilibrium is stable, and A, B, C must be all positive. They are usually denoted by the letters a^2, b^2, c^2. The equation (3) thus becomes

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \ \text{..................(4).}$$

This surface is usually called the ellipsoid of elasticity, and its axes the axes of elasticity. It is assumed that the directions

of these axes and the values of a, b, c are constant throughout the medium. A medium in which a, b, c are all or any of them different is called a crystal. If all are unequal it is called a biaxal crystal. If two of them are equal and the third different it is called a uniaxal crystal.

If the particle be displaced parallel to the axis of x and the other particles be undisturbed it will oscillate in a time $\frac{2\pi}{a}$, for its motion is given by the equation $\frac{d^2u}{dt^2} = -a^2u$.

2. It is then assumed that under these circumstances a particle so displaced will draw an adjacent particle into a precisely similar state of displacement, and that this again will draw the next, and so on; that thus a series of vibrations will be propagated through the medium, the velocity of propagation being connected with the constant a and the wave length by the simple relation

$$\frac{\lambda}{v} = \frac{2\pi}{a} \text{ or } v = \frac{\lambda}{2\pi} \cdot a.$$

For it is supposed that the wave travels over a wave length while one particle performs a complete oscillation.

If the particle be displaced through a space p in a direction inclined at angles (α, β, γ) to the axes of elasticity, the forces on it parallel to the axes are

$$a^2p\cos\alpha,\ b^2p\cos\beta,\ c^2p\cos\gamma,$$

respectively, and the force on it in the direction of displacement will be $$p\,(a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma),$$
and for its motion in that direction we have therefore

$$\frac{d^2p}{dt^2} = -p\,(a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma).$$

If therefore the motion in that direction alone be considered the time of the particle's oscillation will be

$$\frac{2\pi}{\sqrt{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma}};$$

and if a wave of such vibrations can be propagated through the medium its velocity of propagation will as above be proportional to $$\sqrt{a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma}.$$

But if r be the central radius vector of the ellipsoid of elasticity drawn in the direction of this displacement, we have

$$\frac{1}{r^2} = a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma.$$

Hence the velocity of propagation of the wave corresponding to any given direction of displacement, if such a wave exist, is inversely proportional to the central radius vector of the ellipsoid of elasticity drawn in that direction.

3. At this point it will be well to notice the important *assumption* made. The force on any particle is made to depend on its absolute displacement, and is supposed to be the same as if the other particles were undisplaced. It is evident that the real force will depend on the displacement of the particle relative to the surrounding particles, and quite a different equation of motion from that given above will arise. A particular case of the investigation is given in Airy's *Undulatory Theory of Optics*, Art. 103, and the general problem has been discussed by Cauchy. The simplicity and beauty of the Mathematical results of Fresnel's hypothesis probably more than counterbalance, from the point of view of a mathematical student, the possible superior accuracy of the more complicated hypothesis.

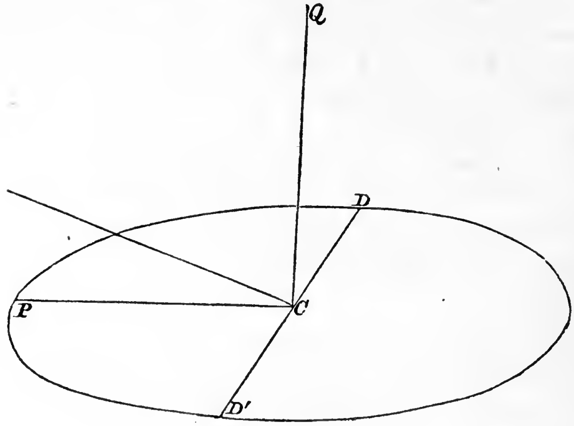
4. In considering the propagation of light through media of any kind, it is necessary to examine not the motion of one particle alone, but to imagine a series of particles simultaneously vibrating similarly. The most simple hypothesis that can be made is that all the similarly displaced particles at any instant lie in a plane, the case ordinarily called a plane wave. It is evident that by the combination of a number of such plane waves we can represent any other form of wave.

A plane wave of light consists of vibrations of the particles of ether in the plane of the wave front, the displacements and velocities of all the particles in that plane being parallel and equal. This wave is propagated with a velocity which in a crystalline medium depends, as above explained, on the direction of the displacement of the particles.

5. The fact that the vibrations which produce light are transversal to the direction of propagation, is deduced from the experimental result that two rays of light polarised in planes at right angles do not interfere. The methods of practically producing polarised light are explained in Airy's *Undulatory Theory.* We assume that polarised light consists of vibrations of the particles of ether in a fixed direction in the plane of the wave front, and that this direction is perpendicular to the plane of polarisation. The former assumption is sufficient to explain the experimental fact, the latter is usually accepted as true.

6. If a series of particles all lying in a plane within a crystalline medium be equally displaced in parallel directions, the force on each of these particles according to Fresnel's hypothesis will not usually be in the direction of displacement, or even in the plane. It may happen however that the resolved part of this force in the plane may coincide with the direction of displacement; and we will prove presently that there are two directions of displacement for which this is the case. If the particles be displaced in either of these directions the force perpendicular to the plane will produce vibrations perpendicular to that plane, which therefore do not produce light; the other parts of the force will cause all the particles to oscillate equally in the plane front, and will thus produce a wave of light, if we assume that the particles oscillating in this plane immediately put in motion those in a contiguous parallel plane. The velocity of propagation of the wave will also, by what has preceded, be inversely proportional to the radius vector of the ellipsoid of elasticity drawn in the direction of the displacement.

7. Suppose that DPD' represents the central section of the ellipsoid of elasticity by a plane parallel to the wave front, and let C be its centre, CP the direction of displacement, CD the diameter of the section conjugate to CP, and CQ the diameter of the ellipsoid conjugate to the plane PCD. Then the force of restitution is perpendicular to the plane QCD, since this is the plane to which CP is conjugate, and if the resolved part of this force in the plane of the wave front coincide with CP, we must have CP and CD at right angles, or CP must be an axis of the section



DPD'. Hence the two directions of vibration with which the resolved part of the corresponding force in the plane coincides are the axes of the section of the ellipsoid of elasticity by the plane front, and the velocities of propagation of the corresponding waves are inversely proportional to the lengths of those axes.

8. If the equation of the plane front at first be

$$lx + my + nz = 0 \quad \text{.....................(1)},$$

and λ, μ, ν, the direction cosines of either axis of the section, we have (Aldis, *Solid Geom.* Art. 56) the equations

$$\left.\begin{array}{r} l\lambda + m\mu + n\nu = 0 \\ \dfrac{l}{\lambda}(b^2 - c^2) + \dfrac{m}{\mu}(c^2 - a^2) + \dfrac{n}{\nu}(a^2 - b^2) = 0 \end{array}\right\} \text{......(2)}$$

to determine λ, μ, ν, the direction cosines of the lines of displacement; and if v be the velocity of propagation of either wave we have to determine v the equation

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \text{...........(3)}.$$

(Aldis, *Solid Geom.* Art. (56), Formula (10), altering a^2, b^2, c^2, r^2 into $\frac{1}{a^2}$, $\frac{1}{b^2}$, $\frac{1}{c^2}$, $\frac{1}{v^2}$ respectively.)

If all the particles in the plane (1) be displaced in any other direction than either of those given by (2), these displacements can be resolved into two, one in each of those directions, and there will then result two sets of vibrations travelling with the velocities given by (3). Hence if at any instant there be a series of particles in the plane (1) vibrating equally in parallel directions, after a unit of time vibrations will be excited in the two planes

$$lx + my + nz = v_1,$$
$$lx + my + nz = v_2,$$

v_1 v_2 being the values of v obtained from (3). Also each of these sets of vibrations will compose a wave of *polarised* light, the planes of polarisation being perpendicular to the two lines whose direction cosines are given by (2).

9. If the envelope of the plane

$$lx + my + nz = v \text{.....................(1)}$$

be investigated, where l, m, n, v are connected by the equations

$$l^2 + m^2 + n^2 = 1 \text{.....................(2)},$$

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0 \text{.....................(3)},$$

we shall obtain the equation of a surface which all the wave fronts touch after a unit of time, in whatever direction the original wave front may have been situated.

Differentiating (1), (2) and (3) we have

$$xdl + ydm + zdn - dv = 0,$$
$$ldl + mdm + ndn = 0,$$

$$\frac{ldl}{v^2 - a^2} + \frac{mdm}{v^2 - b^2} + \frac{ndn}{v^2 - c^2} - vdv \left\{ \frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2} \right\} = 0.$$

Whence, using indeterminate multipliers, we obtain

$$x + Al + \frac{Bl}{v^2 - a^2} = 0 \text{.....................(4)},$$

$$y + Am + \frac{Bm}{v^2 - b^2} = 0 \text{.....................(5)},$$

$$z + An + \frac{Bn}{v^2 - c^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(6)},$$

$$1 + Bv\left\{\frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2}\right\} = 0 \ldots \text{(7)}.$$

Multiplying the first three of these equations by l, m, n respectively and adding, we get,

$$v + A = 0.$$

Transposing the third terms of these same equations, squaring and adding, we get, if $r^2 = x^2 + y^2 + z^2$,

$$r^2 + 2Av + A^2 = B^2\left\{\frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2}\right\};$$

$$\therefore r^2 - v^2 = -\frac{B}{v}, \text{ by (7)};$$

$$\left.\begin{array}{ll} \therefore \text{by (4) } x = vl\left\{1 + \dfrac{r^2 - v^2}{v^2 - a^2}\right\} & = lv\,\dfrac{r^2 - a^2}{v^2 - a^2} \\ \text{(5) } y & = mv\,\dfrac{r^2 - b^2}{v^2 - b^2} \\ \text{(6) } z & = nv\,\dfrac{r^2 - c^2}{v^2 - c^2} \end{array}\right\} \ldots \text{(8)}.$$

Again from (4), (5), and (6) multiplying them by x, y, z respectively and adding

$$r^2 + Av + B\left(\frac{lx}{v^2 - a^2} + \frac{my}{v^2 - b^2} + \frac{nz}{v^2 - c^2}\right) = 0,$$

or

$$r^2 - v^2 + \frac{B}{v}\left\{\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2}\right\} = 0, \text{ by (8)}.$$

Whence, putting for B its value, the equation required becomes

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1.$$

This can be reduced into a different form, for multiplying by $r^2 \equiv x^2 + y^2 + z^2$ it becomes

$$\frac{r^2x^2}{r^2 - a^2} - x^2 + \frac{r^2y^2}{r^2 - b^2} - y^2 + \frac{r^2z^2}{r^2 - c^2} - z^2 = 0,$$

or

$$\frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2} = 0.$$

The equation of the wave surface can also be deduced in the following manner.

The perpendicular on any tangent plane to the surface being inversely proportional to a principal axis of the parallel central section of the ellipsoid of elasticity, it follows that the polar reciprocal of the wave surface with reference to the origin is an *apsidal surface* of this ellipsoid. Whence by Salmon, *Solid Geometry*, Art. 463, the wave surface is also an apsidal surface of the reciprocal surface of the ellipsoid of elasticity, that is of the ellipsoid whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From this property its equation can be easily deduced by eliminating l, m, n between the equation

$$\frac{a^2l^2}{r^2-a^2}+\frac{b^2m^2}{r^2-b^2}+\frac{c^2n^2}{r^2-c^2}=0,$$

which gives the lengths of the axes of the section of (9) by the plane whose equation is $lx+my+nz=0$, and the equations

$$x=lr,\ y=mr,\ z=nr,$$

whence we get

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0,$$

where $$r^2=x^2+y^2+z^2.$$

10. If with the different points of the original wave front as centres we describe a series of equal wave surfaces it is evident that the plane

$$lx+my+nz=v$$

will touch them all. That is, the new wave front may be regarded as the envelope of these wave surfaces. This is analogous to the case of propagation of light through a homogeneous medium, in which case the wave surfaces are spheres. We may also fairly suppose that the point in which the wave surface having any given point of the original wave front as centre touches the second wave front, is the point at which the disturbance in the second wave front is produced by the disturbance at the

given point of the first front, and the line joining these points is the direction of the *ray* proceeding from the first point. A ray must be considered as a small portion of a wave separated from the rest. The existence of such rays must be accepted as a fact; the theoretical explanation of the separation of a portion of a wave from the rest need not be considered here, belonging rather to the question of diffraction.

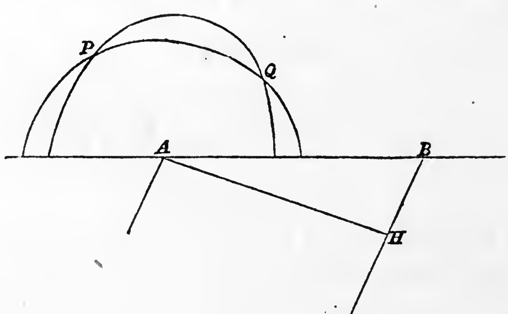
It is not difficult to see that the reciprocal ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

has important properties relating to the *ray velocities* analogous to those which the ellipsoid of elasticity possesses with relation to wave velocities. These the student can develope for himself.

11. If a wave of light be incident from vacuum into a double refracting medium, we may suppose the vibration of each point of the incident wave to produce after a time, a vibration at some point of the wave surface described with the point of incidence as centre.

Let the plane of the paper be the plane of incidence, and let AH be the trace of the front of the wave on the plane of the paper, AB the trace of the face of the crystal. Also let PQ be the



wave surface to some point of which the disturbance produced by A has arrived when the disturbance at H has reached B. The vibrations at intermediate points will have reached points of wave surfaces similar and similarly situated to PQ, but suc-

cessively diminishing in size. Any plane drawn through B perpendicular to the plane of the paper touching the surface PQ will touch all these other surfaces and will be a front of the refracted wave. There can be two such planes drawn, and thus one incident wave will produce two refracted waves. The corresponding refracted rays will be obtained by joining A with the points of contact of these planes with PQ.

12. The preceding Article gives the refracted rays when a ray passes from any homogeneous medium into a double refracting crystal. The following construction applies when a ray passes from any medium into any other.

With the point of incidence of the ray on the common surface of the media as centre, describe in the second medium the half of the wave surface belonging to each medium. Produce the incident ray to cut the surface belonging to the first medium, and at the point of intersection draw a tangent plane. This tangent plane will cut the bounding plane of the media in a straight line. Through this line draw tangent planes to the wave surface of the second medium. The lines joining the point of contact of these tangent planes to the point of incidence of the ray will be the refracted rays.

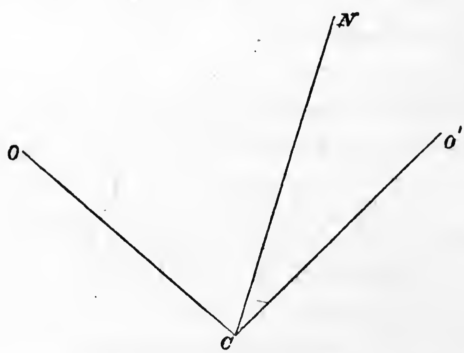
It would appear at first from this construction that a single ray passing from one double refracting medium into another would give rise to four rays, since the incident ray would meet the wave surface of the first medium in two points. We shall see however presently (Art. 15) that if a ray proceeding in any direction within a crystal have originally been refracted from air, it must be polarised in one or other of two definite planes according as it is considered to be proceeding to one or other of the points in which its direction cuts the wave surface; and thus if the given ray be polarised in either of these planes we must only take one of the points as the point to which the incident ray corresponds. If the given ray be either unpolarised or polarised in any other plane it must have arisen from two rays of common light, and must be considered to consist of two rays polarised in the required planes travelling with different velocities. We

should in this case expect four rays, which the construction would give. The construction includes the last article as a particular case.

13. Returning to Art. 8, we see that for all ordinary positions of the wave front, there are two velocities of propagation of the wave. These two will be equal if the wave front coincide with one of the circular sections of the ellipsoid of elasticity, and in that case, whatever be the direction of the vibrations in the plane of the front, only one wave will be propagated. The two lines perpendicular to these positions of the wave front are called the optic axes of the crystal, or the lines of *equal wave velocity.*

The planes of polarisation of the two rays corresponding to any given wave front are connected with the optic axes by a very simple relation, which we will now investigate.

Let CN be the normal to the wave front, CO, CO' the optic axes of the crystal. Then the planes of polarisation of the two rays are planes which contain CN and the axes of the sec-



tion of the ellipsoid of elasticity by a plane perpendicular to CN. This section will evidently cut the circular section perpendicular to CO in a line perpendicular to the plane OCN. Similarly it will cut the other circular section in a line perpendicular to the plane $O'CN$. Hence the radii of the section by

the wave front perpendicular to the planes OCN, $O'CN$ are equal and therefore they are equally inclined to the axes of the section. The planes of polarisation of the two rays are therefore planes through CN bisecting the angles between the planes OCN and $O'CN$.

14. Again let v_1, v_2 be the velocities of the two waves corresponding to the same wave front. We can express these velocities in terms of the angles OCN, and $O'CN$, as follows.

The equation of the ellipsoid of elasticity being

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \quad \text{..................} \quad (1),$$

the equations of the planes of circular section are

$$x\sqrt{a^2 - b^2} \pm z\sqrt{b^2 - c^2} = 0 \quad \text{..............} \quad (2),$$

and that of the wave front is

$$lx + my + nz = 0 \quad \text{....................} \quad (3).$$

Hence if we denote the angles OCN, $O'CN$ by θ, θ' respectively, we have

$$\left.\begin{aligned} \cos\theta &= \frac{l\sqrt{a^2 - b^2} - n\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}} \\ \cos\theta' &= \frac{l\sqrt{a^2 - b^2} + n\sqrt{b^2 - c^2}}{\sqrt{a^2 - c^2}} \end{aligned}\right\} \quad \text{.........} \quad (4),$$

$$\left.\begin{aligned} \therefore\ (\cos\theta' + \cos\theta)\sqrt{a^2 - c^2} &= 2l\sqrt{a^2 - b^2} \\ (\cos\theta' - \cos\theta)\sqrt{a^2 - c^2} &= 2n\sqrt{b^2 - c^2} \end{aligned}\right\} \quad \text{......} \quad (5).$$

Again v_1, v_2 are the roots of the equation

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0.$$

Hence

$$\begin{aligned} v_1^2 + v_2^2 &= l^2(b^2 + c^2) + m^2(c^2 + a^2) + n^2(a^2 + b^2) \\ &= a^2 + c^2 - l^2(a^2 - b^2) + n^2(b^2 - c^2) \quad \text{since } l^2 + m^2 + n^2 = 1 \\ &= a^2 + c^2 - (a^2 - c^2)\cos\theta\cos\theta' \text{ by (4)} \quad \text{..............} \quad (6). \end{aligned}$$

$$\begin{aligned} v_1^2v_2^2 &= l^2b^2c^2 + m^2c^2a^2 + n^2a^2b^2, \\ &= a^2c^2 - c^2l^2(a^2 - b^2) + n^2a^2(b^2 - c^2)\,; \end{aligned}$$

$$\therefore 4v_1^2v_2^2 = 4a^2c^2 - (a^2-c^2)\{c^2(\cos\theta'+\cos\theta)^2 - a^2(\cos\theta'-\cos\theta)^2\} \text{ by (5)},$$

$$= 4a^2c^2 + (a^2-c^2)^2(\cos^2\theta+\cos^2\theta') - 2(a^4-c^4)\cos\theta\cos\theta' \ldots (7).$$

Hence squaring (6) and subtracting (7) we get

$$(v_1^2 - v_2^2)^2 = (a^2-c^2)^2\{1-\cos^2\theta - \cos^2\theta' + \cos^2\theta\cos^2\theta'\},$$

$$= (a^2-c^2)^2 \sin^2\theta \sin^2\theta';$$

$$\therefore v_1^2 \sim v_2^2 = (a^2-c^2)\sin\theta\sin\theta' \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

From (6) and (8) we easily deduce by adding and subtracting

$$\left.\begin{aligned} v_1^2 &= a^2\sin^2\frac{\theta\pm\theta'}{2} + c^2\cos^2\frac{\theta\pm\theta'}{2} \\ v_2^2 &= a^2\sin^2\frac{\theta\mp\theta'}{2} + c^2\cos^2\frac{\theta\mp\theta'}{2} \end{aligned}\right\} \ldots\ldots (9).$$

The results of equations (8) and (9) are easily seen to coincide with those deduced in a different manner in Salmon's *Solid Geometry*, Art. 245. Analogous results can be obtained for *ray velocities* from the reciprocal ellipsoid. (Lloyd, *Wave Theory of Light*, Art. 186.)

15. The formulæ of the last article enable us to determine completely the circumstances of the vibrations of the two rays corresponding to the same wave front in the crystal. They do not however determine the plane of polarisation if we are only given the direction in which the *ray* proceeds within the crystal. For this purpose we must revert to the wave surface of Art. 9.

Let a ray meet the wave surface at the point x, y, z, let l, m, n be the direction cosines of the normal to the wave front to which the ray belongs, and λ, μ, ν the direction cosines of the direction of vibration of the particles in the ray. Then we have, if v be the corresponding wave velocity,

$$v^2 = a^2\lambda^2 + b^2\mu^2 + c^2\nu^2,$$

where $$\frac{l}{\lambda}(b^2-c^2) + \frac{m}{\mu}(c^2-a^2) + \frac{n}{\nu}(a^2-b^2) = 0,$$

and $$l\lambda + m\mu + n\nu = 0.$$

Whence eliminating n we get

$$l\left\{\frac{\nu}{\lambda}(b^2-c^2)-\frac{\lambda}{\nu}(a^2-b^2)\right\}+m\left\{\frac{\nu}{\mu}(c^2-a^2)-\frac{\mu}{\nu}(a^2-b^2)\right\}=0,$$

or
$$\frac{l}{\lambda}\{b^2-\lambda^2a^2-\mu^2b^2-\nu^2c^2\}-\frac{m}{\mu}\{a^2-\nu^2c^2-\mu^2b^2-\lambda^2a^2\}=0,$$

$$\therefore \frac{l}{\lambda(a^2-v^2)}=\frac{m}{\mu(b^2-v^2)}=\frac{n}{\nu(c^2-v^2)} \text{ by symmetry.}$$

These equations determine λ, μ, ν in terms of v.

Combining these results with the equations (8) of Art. (9), we easily obtain

$$\frac{\lambda(r^2-a^2)}{x}=\frac{\mu(r^2-b^2)}{y}=\frac{\nu(r^2-c^2)}{z},$$

which give the direction of vibration in the ray proceeding to any given point (x, y, z).

A geometrical interpretation can be given to these equations. The co-ordinates of the foot of the perpendicular on the tangent plane to the wave surface at x, y, z are with our previous notation, lv, mv, nv, and the direction cosines of the line joining this point with the point of contact are proportional to

$$x-lv,\ y-mv,\ z-nv.$$

But we have by equations (8) of Art. (9),

$$lv=x\frac{v^2-a^2}{r^2-a^2};$$

$$\therefore x-lv=\frac{x(r^2-v^2)}{r^2-a^2}.$$

Similarly
$$y-mv=\frac{y(r^2-v^2)}{r^2-b^2},$$

$$z-nv=\frac{z(r^2-v^2)}{r^2-c^2}.$$

Hence λ, μ, ν are proportional to $x-lv$, $y-mv$, $z-nv$, or the direction of the vibration constituting any ray is the projection of the ray on the tangent plane to the wave surface at the point

where it meets it. The plane of polarisation is of course perpendicular to this. The plane of polarisation of a ray proceeding in a double refracting medium is usually taken to be the plane containing the normal to the wave front and perpendicular to the direction of vibration.

This result may be otherwise obtained. If λ, μ, ν are the direction cosines of the direction of displacement of a particle, those of the resultant force are proportional to $a^2\lambda$, $b^2\mu$, $c^2\nu$. Hence the direction of displacement is perpendicular to the tangent plane drawn to the reciprocal ellipsoid at the point where the line of the resultant force meets it. From this, since the wave surface is the apsidal surface of the reciprocal ellipsoid, it follows by Arts. 461, 462 of Salmon's *Solid Geometry* that the direction of displacement, the direction of the resultant force, the normal to the wave front and the ray all lie in one plane. Hence the direction of displacement is the projection of the radius vector on the tangent plane to the wave surface.

It also follows that the ray and the direction of the resultant force are at right angles.

16. The wave surface has thus been shown to possess the following properties.

Its tangent planes give the positions of the wave fronts after a given time, and the perpendiculars on those tangent planes from the centre represent the *velocities of wave fronts* in different directions.

Its radii vectores from the centre to the points of contact are the directions of the rays corresponding to the different wave fronts, and the lengths of these radii vectores represent the corresponding *ray velocities.*

The projections of these radii on the tangent planes give the directions of the vibrations in the corresponding rays.

17. The equation of the wave surface becomes by multiplying up

$$x^2(r^2-b^2)(r^2-c^2)+y^2(r^2-c^2)(r^2-a^2)+z^2(r^2-a^2)(r^2-b^2)$$
$$=(r^2-a^2)(r^2-b^2)(r^2-c^2).$$

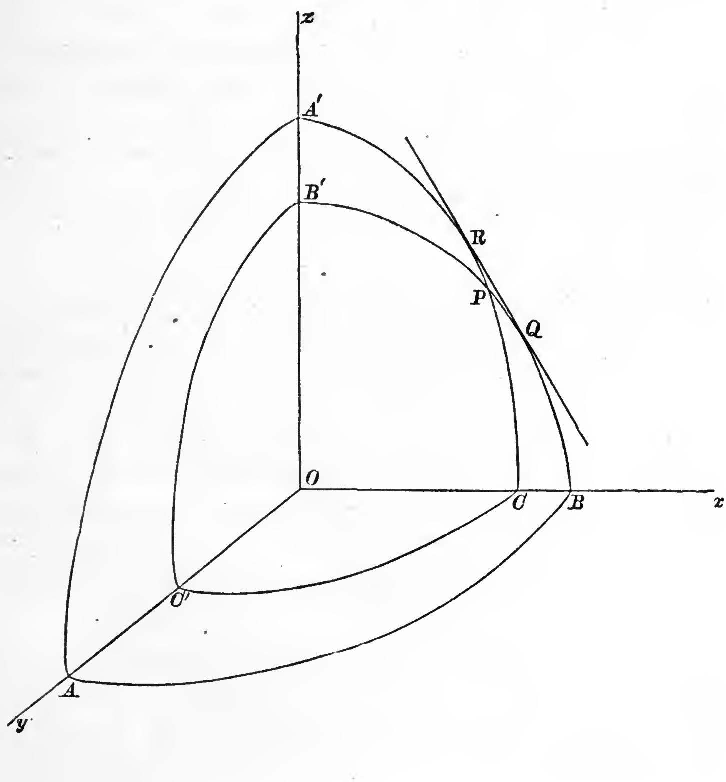
Its trace on the plane of xy is obtained by putting $z=0$; we then get

$$(x^2+y^2-c^2)\{x^2(x^2+y^2-b^2)+y^2(x^2+y^2-a^2)\}$$
$$=(x^2+y^2-a^2)(x^2+y^2-b^2)(x^2+y^2-c^2)$$

whence $$x^2+y^2=c^2,$$

or $$a^2x^2+b^2y^2=a^2b^2.$$

That is, the trace consists of a circle whose radius is c and an ellipse whose axes are $2b$ and $2a$ in the directions Ox and Oy respectively. Similarly the trace on the plane of yz consists of a circle whose radius is a and an ellipse whose axes are $2c$ and $2b$; and the trace on the plane of zx, of a circle of radius b and an ellipse whose axes are $2a$ and $2c$.



The figure represents these traces on the supposition that a, b, c are in descending order of magnitude.

The figure shows that the two curves in the plane of zx cut at a point P whose co-ordinates can be obtained from the two equations

$$x^2 + z^2 = b^2,$$

$$a^2x^2 + c^2z^2 = a^2c^2.$$

Whence $$x^2 = c^2\frac{a^2 - b^2}{a^2 - c^2}, \quad z^2 = a^2\frac{b^2 - c^2}{a^2 - c^2}.$$

It will be found that these values combined with $y = 0$ satisfy the conditions for a singular point on the surface, for they make $\frac{dF}{dx}$, $\frac{dF}{dy}$ and $\frac{dF}{dz}$ vanish, where $F(x, y, z) = 0$ is the equation of the wave surface. There is therefore at the point P a tangent cone whose equation can be easily obtained (Aldis, *Solid Geometry*, Art. 77). There is also a corresponding normal cone of which OP is evidently one generating line. The line OP is known as the line of *single ray velocity*. It is perpendicular to a circular section of the reciprocal ellipsoid.

From this it follows that there is an infinite number of incident rays which may give rise to a ray within the crystal in the direction OP, since a ray in that direction may belong to any one of the wave fronts to which the generating lines of the above normal cone are perpendicular. The plane of polarisation of the ray considered as belonging to any particular wave front will, by Art. 15, be perpendicular to the plane through OP and the normal to the wave front. The direction of the incident ray which is required to produce the ray belonging to each wave front can be determined by Arts. 11 or 12. It is tolerably evident that all such rays will lie on a conical surface.

If then we contrive to make a pencil of rays converge to a point on the surface of a plate of biaxal crystal in such a manner that the converging pencil shall include all the required rays, and by some means limit the direction in which light can pass

through the crystal from the point of incidence, to the direction OP, there will be an assemblage of rays proceeding all in that one direction. These rays, on arriving at the other face of the crystal, will be all differently refracted, since they correspond to different wave fronts, and there will thus be produced on emergence a hollow diverging cone of rays, each ray of the cone having a different plane of polarisation.

This is found to be experimentally the fact. The limitation that the rays shall only pass in one direction, is effected by placing two thin plates of metal, with a small hole perforated in each, in such positions on the two sides of the plate of crystal that the line joining them shall coincide with the line of single ray velocity, a pencil of light is then made to converge to a point at the hole in one of the pieces of metal and the light received on a screen on the other side of the plate. The size of the ring of light formed at different distances and the polarisation of each ray are found to agree with theory. This phenomenon is known by the name of *external conical refraction.*

18. It is evident again from the last figure that a common tangent can be drawn to the two curves of section in the plane zx, and a plane through this tangent and perpendicular to the plane zx will touch the surface in two points Q and R. It can be shown however that this plane really touches the surface along a curve, and that this curve is a circle of which RQ is the diameter. (Salmon's *Solid Geometry*, Art. 465.)

The equation of this plane is easily obtained, for since QR touches both the circle and ellipse, its equation must assume either of the forms

$$lx + nz = b\sqrt{l^2 + n^2},$$

or
$$lx + nz = \sqrt{l^2c^2 + n^2a^2}.$$

Whence
$$l^2c^2 + n^2a^2 = (l^2 + n^2)\, b^2:$$

$$\therefore\ l^2(b^2 - c^2) = n^2(a^2 - b^2) \text{ or } \frac{l^2}{a^2 - b^2} = \frac{n^2}{b^2 - c^2} = \frac{l^2 + n^2}{a^2 - c^2}.$$

Hence the plane through QR perpendicular to the plane zx has for its equation

$$x\sqrt{a^2-b^2} \pm z\sqrt{b^2-c^2} = b\sqrt{a^2-c^2} \ldots\ldots\ldots (1).$$

It is therefore perpendicular to an optic axis, which is also *a priori* evident, since the rays from O to Q and R correspond to the same wave front.

The equations (8) of Art. 9 give the co-ordinates of the point of contact of any tangent plane

$$lx + my + nz = v.$$

For the co-ordinates of the point of contact of (1), confining ourselves to the upper sign and putting

$$l = \sqrt{\frac{a^2-b^2}{a^2-c^2}},\quad m = 0,\quad n = \sqrt{\frac{b^2-c^2}{a^2-c^2}},\quad v = b,$$

the second of those equations reduces to an identity, and the first and third give

$$x^2 + y^2 + z^2 - a^2 = -\frac{\sqrt{(a^2-b^2)(a^2-c^2)}}{b}\,x,$$

$$x^2 + y^2 + z^2 - c^2 = \frac{\sqrt{(b^2-c^2)(a^2-c^2)}}{b}\,z.$$

Which two spheres by their intersection determine a circle at every point of which the plane (1) touches the wave surface.

19. If therefore a ray of light be incident on a biaxal crystal in such a direction that the front of the refracted wave shall be the plane (1) of the last article, there will be produced a cone of rays proceeding to all points of the circle of contact, the planes of polarisation of all these rays being different, the direction of vibration of the ether in any ray being parallel to the chord of the circle joining its extremity with Q, by Art. 15. Hence the plane of polarisation of the ray at Q is the plane zx, while that of the ray at R is perpendicular to the plane zx. The plane of polarisation therefore turns through a right angle

while the point of incidence of the ray sweeps half round the circle of contact. All these rays when incident on the second face of the plate of crystal will emerge parallel, since they all belong to one wave front, and we shall thus obtain a hollow cylinder of light.

The limitation of the direction of light is made by placing two plates of metal with a small hole in each, one of them being in contact with the crystal, the other at some distance from it, and allowing only the ray which has passed through both these to enter the crystal. If the line joining these be experimentally adjusted to the right position, a cylindrical pencil of rays will be found to issue from the plate. This cylinder is very small unless the piece of crystal be of considerable thickness. The phenomenon may be made more conspicuous by receiving the light on a lens of short focal length which will convert the cylinder into a hollow cone of light which may either be received on a screen or by the eye. If this pencil be viewed through a Nichol's prism, the polarisation of its different rays is found to agree with the theory. This phenomenon is known as *internal conical refraction.*

20. The preceding investigations apply to the most general case of double refraction, namely that in which all three constants a, b, c are unequal. If all three become equal, the ellipsoid of elasticity becomes a sphere, the wave surface two coincident spheres, and double refraction ceases. If however only two are equal, as a and b, double refraction will still exist. In this case the ellipsoid of elasticity becomes a spheroid, and its two sets of circular sections coincide. There is thus only one optic axis, which is called the axis of the crystal, and such crystals are, as before stated, called uniaxal crystals.

The equation (3) of Art. 8

$$l^2(v^2-b^2)(v^2-c^2)+m^2(v^2-c^2)(v^2-a^2)+n^2(v^2-a^2)(v^2-b^2)=0,$$

becomes

$$(v^2-a^2)\{v^2-(l^2+m^2)c^2-n^2a^2\}=0,$$

whence

$$v^2=a^2, \text{ or } v^2=a^2n^2+c^2(l^2+m^2).$$

So that of the two waves corresponding to a given direction of front, one has a constant velocity a, and the other has a velocity

$$\sqrt{a^2 \cos^2 \theta + c^2 \sin^2 \theta},$$

where θ is the angle between the normal to the wave front and the axis of the crystal.

The equation of the wave surface reduces to the form

$$x^2 (r^2-a^2)(r^2-c^2) + y^2 (r^2-a^2)(r^2-c^2) + z^2 (r^2-a^2)^2 = (r^2-c^2)(r^2-a^2)^2,$$

whence

$$r^2 - a^2 = 0, \text{ or } x^2 + y^2 + z^2 = a^2,$$

or
$$(x^2 + y^2)(r^2 - c^2) + z^2 (r^2 - a^2) = (r^2 - a^2)(r^2 - c^2),$$

which reduces to

$$a^2 (x^2 + y^2) + c^2 z^2 = a^2 c^2.$$

Hence the wave surface reduces to a sphere and a spheroid.

The formula of Art. 15 will still give the direction of vibration of the ether for the ray which proceeds to a point on the spheroid, which is usually called the extraordinary ray. This direction will easily be seen to lie in a plane containing the axis of z, and the plane of polarisation of this ray is thus perpendicular to this plane. The direction of vibration of the ray which proceeds to a point of the sphere will be easily seen to be perpendicular to the plane through the ray and the axis of z, and this plane is therefore its plane of polarisation.

21. Fresnel's Theory, of which we have endeavoured to explain the main features, is undoubtedly not a sound *dynamical Theory.* It has however the great merit of representing accurately the facts of double refraction as far as experiment at present has tested them, and in one instance has led to the discovery of facts (the conical refractions) previously unobserved. It is also probably better adapted for students than any other of the theories yet promulgated. Those who take an interest in these theories can refer to Cauchy's *Exercices de Mathématiques*, Tomes III. and IV., MacCullagh's papers in the *Transactions of the Royal Irish Academy,* and Green's papers in Vol. VII. of the *Cambridge Philosophical Transactions.* A full account and comparison of these and other theories will be found in the

Report to the British Association, in 1862, by one of the highest authorities on the subject, Professor Stokes. After careful examination of them all he expresses his own opinion that the true dynamical theory of Double Refraction has yet to be found.

Probably when the Newton of Physical Optics has succeeded in linking together all the phenomena of Light into one continuous chain, the name of Fresnel will yet be remembered with a reverence akin to that which astronomers feel for Copernicus and Kepler.

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